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Problems and hopes in nonsymmetric gravity

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Abstract
We consider the linearized nonsymmetric theory of gravitation (NGT) within the background of an expanding universe and near a Schwarzschild mass. We show that the theory always develops instabilities unless the linearized nonsymmetric Lagrangian reduces to a particular simple form. This form contains a gauge-invariant kinetic term, a mass term for the anti-symmetric metric field and a coupling with the Ricci curvature scalar. This form cannot be obtained within NGT. Based on the linearized Lagrangian we know to be stable, we consider the generation and evolution of quantum fluctuations of the anti-symmetric gravitational field (B-field) from inflation up to the present day. We find that a B-field with a mass $m \propto 0.03(H_I/10^{13}\ \mathrm{GeV})^4$ eV is an excellent dark matter candidate.

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1. Introduction

While Einstein's general relativity (GR) has stood all direct experimental tests [13], there are also reasons to try to extend GR. For example, the mysterious nature of dark energy and dark matter might become resolved within a modified theory of gravity.

Another reason to try to extend GR is the notion of generality. Within the framework of GR torsion is not included in a natural, geometric way. Indeed, any calculation of the connection (either by requiring metric compatibility or by using the first-order formalism) leads to the (symmetric) Levi-Cività connection. One is then free to add torsion, but torsion does not follow naturally from the theory. An interesting generalization of GR would generate torsion in a purely geometric way, analogous to the way the Levi-Cività connection is generated in GR.

The nonsymmetric gravitational theory (NGT) [7] is an extension of GR that drops the standard axiom of GR that the metric is a symmetric tensor. Thus, we decompose the general, nonsymmetric metric $g_{\mu\nu}$ in its symmetric and anti-symmetric parts,

$$g_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu} = g_{(\mu\nu)}$, $B_{\mu\nu} = g_{[\mu\nu]}$ and $(\cdot)$ and $[\cdot]$ indicate normalized symmetrization and anti-symmetrization, respectively. Indeed, there is no physical principle that tells us that the metric should be symmetric and therefore such a generalization is very interesting to study.

Indeed, the extra structure of NGT produces interesting results on the issues of dark energy and dark matter [8–10] and it will also be clear that such a theory produces torsion in a very natural way. Unfortunately, the nonsymmetric theory of gravitation suffers from all kinds of problems. The first main problem is the non-uniqueness of the theory, as described in [5]. Since torsion is available and since the linearization procedure is not unambiguous, the final linearized Lagrangian is (degenerately) determined by 11 free parameters. The second problem, as described in [4], is the possibility of propagating ghost modes. Fortunately, this problem can be relatively easily solved by the introduction of a mass term for the B-field [3, 7].

In this paper, we consider NGT linearized around a GR configuration. By explicitly constructing two different backgrounds (FLRW universe and Schwarzschild), we show that the evolution of the B-field is unstable. By considering the most general form of the linearized Lagrangian, we can explicitly point out which terms cause these instabilities. In [5], it is shown that both these terms cannot be removed and that these terms are not a relic of the linearization. Based on this analysis, we are able to write a consistent, stable linearized Lagrangian for the B-field. We next canonically quantize the B-field in inflation and follow its dynamics in radiation and matter era. This analysis shows that the B-field is an excellent dark matter candidate, provided the mass is of the order of the neutrino masses.

2. The linearized Lagrangian

Since GR is very successful, it is natural to assume that any modification of the theory should be relatively small. Therefore, we consider NGT in the limit of a small B, but an arbitrary G. The linearization of the full, general Lagrangian is done in appendix A of [5]. The result is

$$\mathcal{L} = \sqrt{-G}\big[R + 2\Lambda - \tfrac{1}{12}H^2 + \big(\tfrac{1}{4}m^2 + \beta R\big)B^2 \\ - \alpha R_{\mu\nu}B^{\mu\alpha}B_{\alpha}{}^{\nu} - \gamma R_{\mu\alpha\nu\beta}B^{\mu\nu}B^{\alpha\beta}\big] + \mathcal{O}(B^3). \tag{2}$$

Here, the curvature terms $R_{\mu\alpha\nu\beta}$, $R_{\mu\nu}$ and R all refer to the GR background. $H_{\mu\nu\rho}$ is the field strength associated with $B_{\mu\nu}$. The coefficients α, β and γ are determined by the parameters of the 'full' Lagrangian and the unambiguous decomposition of the metric in its symmetric and anti-symmetric parts. It is important to note that one cannot consistently choose the parameters of the full theory in such a way that $\gamma = 0$ (see appendix A of [5]). The parameters α and β can in principle be set to zero; however *a priori* there is no reason to do this. A mass is naturally generated in the presence of a nonzero cosmological constant and in fact one has

$$\tfrac{1}{4}m^2 = \Lambda\big(\tfrac{1}{2} - \rho + 4\sigma\big) \propto 10^{-84}\ \mathrm{GeV}^2, \tag{3}$$

where we assume that the parameters ρ and σ are of order unity. ρ and σ depend on the same parameters as α, β and γ in (2) (see appendix A of [5]). Note that the inequality in (3) is not necessarily true at all times, since the cosmological term may change during the evolution

Table 1. The scale factor and conformal time in different eras.

Era	a	η
de Sitter inflation	$a = -\frac{1}{H_I \eta}$	$\eta \leqslant -\frac{1}{H_I}$
Radiation	$a = H_I \eta$	$\frac{1}{H_I} \leqslant \eta \leqslant \eta_{\rm eq}$
Matter	$a = \frac{H_I}{4\eta_{\rm eq}^2}(\eta + \eta_{\rm eq})^2$	$\eta \geqslant \eta_{\rm eq}$

of the universe (for example during phase transitions). The field equations derived from the Lagrangian (2) are

$$(\sqrt{-G})^{-1}\tfrac{1}{2}\partial_\rho(\sqrt{-G}H^{\rho\mu\nu}) + \left(\tfrac{1}{2}m^2 + 2\beta R\right)B^{\mu\nu} - \alpha\left(B^{\nu\alpha}R^{\mu}{}_{\alpha} + B^{\alpha\mu}R^{\nu}{}_{\alpha}\right) - 2\gamma B^{\alpha\beta}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta} + \mathcal{O}(B^2) = 0 \tag{4}$$

$$R_{\mu\nu} - \tfrac{1}{2}RG_{\mu\nu} - \Lambda G_{\mu\nu} + \mathcal{O}(B^2) = 0. \tag{5}$$

We see that to this order the field equations decouple and it makes sense to consider the symmetric background to be just a GR background. The theory then reduces to an anti-symmetric tensor field coupled to GR.

3. Instabilities in NGT

We first focus on the dynamics of the B-field in an expanding universe[1] [11]. Our background metric is given by the (conformal) Friedmann–Lemaitre–Robertson–Walker metric (FLRW):

$$G_{\mu\nu} = a(\eta)^2\eta_{\mu\nu}, \tag{6}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, η is the conformal time and $a(\eta)$ is the conformal scale factor. The conformal time is related to the standard cosmological time by $a\,\mathrm{d}\eta = \mathrm{d}t$. The scale factor during the different cosmological eras is given in table 1, where $H_I \sim 10^{13}$ GeV is the Hubble parameter during inflation and $\eta_{\rm eq}$ is the conformal time at matter–radiation equality.

For the following discussion, we focus on the 'electric' mode of the B-field: $E_i \equiv B_{0i}$. (the 'magnetic' mode turns out not to be very interesting for our present purpose). If we evaluate the Lagrangian (2) and the field equations (4) in the FLRW background, we find the following equation of motion:

$$\left[\partial_0\partial_0 - \frac{\mathcal{Y}}{\mathcal{X}}\delta^{ij}\partial_i\partial_j + M_{\rm eff}^2\right]\tilde{E} = 0, \tag{7}$$

where

$$E = \frac{\sqrt{\mathcal{Y}}}{\mathcal{X}}\tilde{E}, \tag{8}$$

and the effective mass term is given by

$$M_{\rm eff}^2 = -2\mathcal{Y}a^2 + \frac{\mathcal{Y}''}{2\mathcal{Y}} - \frac{3(\mathcal{Y}')^2}{4\mathcal{Y}^2}. \tag{9}$$

[1] This section is based on [5].

Furthermore, we have defined

$$\mathcal{X} = a^{-2}\big((12\beta + 2\alpha)\mathcal{H}^2 + (12\beta + 4\alpha - 2\gamma)\mathcal{H}' - \tfrac{1}{2}m^2a^2\big) \tag{10}$$

$$\mathcal{Y} = a^{-2}\big((12\beta + 4\alpha - 2\gamma)\mathcal{H}^2 + (12\beta + 2\alpha)\mathcal{H}' - \tfrac{1}{2}m^2a^2\big) \tag{11}$$

and

$$\mathcal{H} = \frac{a'}{a}, \tag{12}$$

where a prime indicates a derivative with respect to conformal time. We see from (7) that $\tilde{E}$ behaves just as a massive vector field, *as long as* $\mathcal{Y}/\mathcal{X} > 0$. On the other hand, if $\mathcal{Y}/\mathcal{X} < 0$ we see that the spatial derivatives appear with the 'wrong' sign. Since in Fourier space these derivatives generate a term proportional to minus the momentum squared, we see that a wrong sign will lead to an exponential growth of the field. Large momenta are no longer suppressed and thus the field will grow without bounds. One could worry about the cases when $M_{\rm eff}^2 < 0$. However on dimensional grounds, the effective mass squared scales in the worst case as $1/\eta^2$. Such a scaling results in a standard power-law enhancement on super-Hubble scales [11] and presents no problem.

3.1. Instabilities during radiation era

In de Sitter inflation $\mathcal{Y}/\mathcal{X} = 1$, and thus the field dynamics are completely regular. However, during radiation era we obtain

$$\left[\partial_0\partial_0 - \frac{H_I^2 m^2\eta^4 + 4(\gamma - \alpha)}{H_I^2 m^2\eta^4 - 4(\gamma - \alpha)}\delta^{ij}\partial_i\partial_j + M_r^2\right]\tilde{E}_r = 0. \tag{13}$$

Here, M_r is the effective mass during radiation, whose precise form is not important for us. We see, however, that we might have problems with the sign of the coefficient in front of the spatial derivatives. For example, if we look at the beginning of radiation era ($\eta = 1/H_I$) we see that if we want $\mathcal{Y}/\mathcal{X}$ to be positive, we need that $m^2\big/H_I^2$ is *at least* $\mathcal{O}(\alpha - \gamma)$. In other words, we approximately need

$$m \geqslant |\alpha - \gamma| H_I \sim |\alpha - \gamma| \times 10^{13}\ \text{GeV}, \tag{14}$$

which, unless $|\alpha - \gamma|$ is very small, contradicts equation (3). Therefore, if we require $\mathcal{Y}/\mathcal{X}$ to be positive, we could drop the purely geometric origin of the Lagrangian and add by hand a large (10^{13} GeV) mass for the B-field, we could fine tune α or γ such that $\alpha - \gamma$ is sufficiently small to satisfy the bound (14) or we could use the more natural requirement that $\alpha = \gamma$. On theoretical grounds only the last of these solutions is satisfactory. A big problem with the first solution is that, while we can always find a mass where the evolution of the mode is stable, we can then also think of more extreme situations where the mode once again becomes unstable. Therefore, we conclude that a natural theory should have $\alpha = \gamma$.

We have also investigated matter era and power-law inflation and we find that similar instabilities are present. However also in these cases $\alpha = \gamma$ stabilizes the system.

3.2. Instabilities around a Schwarzschild mass

We have done a similar analysis in a Schwarzschild background. While the background possesses spherical symmetry, there is no reason to impose this symmetry on the B-field. We will not give any details here (these are given in section 4 of [5]), but will only mention that similar instabilities as in the previous section are present. The unstable modes are precisely the modes that are not spherically symmetric (these modes are not taken into account in [1], where it was claimed that the B-field in a Schwarzschild background is stable). In this case, the requirements for a stable system are either

$$\gamma = 0 \tag{15}$$

or

$$m^2 > \frac{4\gamma G_N \hbar^2}{c^4}\frac{M_0}{r_0^3} \quad (\mathrm{kg}^2), \tag{16}$$

where we explicitly plugged back factors of c, h and G_N. M_0 is the mass of the object we are considering and r_0 is the distance where we require stability. For γ order 1 this requires, e.g., for the exterior of a neutron star ($M_0 \propto M_{\mathrm{sun}}$ and $r_0 \propto 20$ km):

$$m \gtrsim \sqrt{|\gamma|} \times 10^{-19}\ \mathrm{GeV}. \tag{17}$$

However, on theoretical grounds, it is more appealing to require that the B-field is stable for all values of M_0 and r_0. This can only be achieved if we choose $\gamma = 0$. However, as noted in section 2, this choice is not possible within our linearization of NGT.

4. Anti-symmetric metric field as dark matter

Based on the previous section, we know that the only consistent linearized Lagrangian for the B-field is

$$\mathcal{L} = \sqrt{-G}\big[R + 2\Lambda - \tfrac{1}{12}H^2 + \big(\tfrac{1}{4}m^2 + \beta R\big)B^2\big]. \tag{18}$$

While this Lagrangian is not obtainable in NGT, we like to stress that our linearization procedure of NGT lacks any guiding principle (which is reflected in the non-uniqueness of the theory). The analysis of the previous section shows that if we want to make sense of nonsymmetric gravity we need to find a guiding principle that, upon linearization, leads to (18). The most natural way in which this could happen is that by some symmetry in the new theory the dangerous terms in (2) are forbidden. If this is the case, this symmetry would also forbid these terms to be generated by quantum corrections, that otherwise might destabilize the theory. For now we do not know this principle, but we can still study (18). In this section[2], we consider the generation and evolution throughout the cosmological history of quantum fluctuations of the B-field. In particular, we only consider the longitudinal degrees of freedom of the 'magnetic' component [10–12],

$$B_{ij} \equiv -\epsilon_{ijk} B_k, \tag{19}$$

since this mode gives the dominant contribution to the energy density in the limit $m \to 0$. For simplicity we take $\beta = 0$, but keep the mass arbitrary. Indeed, since we lack a guiding principle, (3) does not have to be true. Therefore, we allow the presence of a 'bare' mass for the B-field. In order to quantize the field, we perform a Fourier transformation

$$B^L(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\big[\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} B^L(\eta,\vec{k}) b_{\vec{k}} + \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}} B^{L\star}(\eta,\vec{k}) b^\dagger_{\vec{k}}\big], \tag{20}$$

where η is once again conformal time as given in table 1, with canonical commutation relations

$$\big[b_{\vec{k}}, b^\dagger_{\vec{k}'}\big] = (2\pi)^3\delta^3(\vec{k} - \vec{k}'). \tag{21}$$

During de Sitter inflation we find that the mode functions approach the conformal vacuum

$$B^L_{\mathrm{inf}} \propto \frac{1}{\sqrt{2k}}\,\mathrm{e}^{-\mathrm{i}k\eta} + \mathcal{O}\left(\frac{m^2}{H_I^2}\right). \tag{22}$$

[2] Based on [12].

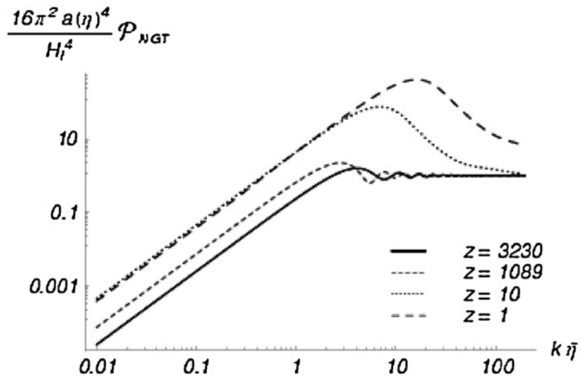

Figure 1. Snapshot of the power spectrum for $mH_I\eta_{\rm eq}^2 = 10^{-2}$.

During radiation era the field equations are solved by

$$B^L_{\rm rad} = \frac{1}{\sqrt{2k}}\left[\alpha_{\vec{k}}\left(1-\frac{\rm i}{k\eta}\right){\rm e}^{-{\rm i}k\eta} + \beta_{\vec{k}}\left(1+\frac{\rm i}{k\eta}\right){\rm e}^{{\rm i}k\eta}\right] + \mathcal{O}\left(\frac{m^2}{H_I^2}\right) \qquad (23)$$

with the Wronskian condition that

$$|\alpha_{\vec{k}}|^2 - |\beta_{\vec{k}}|^2 = 1 \qquad (24)$$

and we choose α and β such that the solutions match at the inflation–radiation transition. Unfortunately, we cannot analytically solve the equations of motion in matter era, so there we need to use numerical analysis. We are interested in the power spectrum, which is given by [10]

$$P_B(\vec{k},\eta) = \frac{H_I^4}{4\pi^2 a^4}\left[\left|\partial_\eta B^L_{\vec{k}}(\eta) + \frac{a'}{a}B^L_{\vec{k}}(\eta)\right|^2 + (k^2 + a^2m^2)\left|B^L_{\vec{k}}\right|^2\right]. \qquad (25)$$

A snapshot of this power spectrum, during matter era, for different redshifts is given in figure 1. We find that at late times the power spectrum becomes dominated by a characteristic peak. This peak is caused by modes that are superhorizon ($k\eta \lesssim 1$) at equality ($z = 3230$), but start to scale as nonrelativistic matter ($\propto a^{-3}$) in matter era and enter the horizon. Modes on small enough scales ($k\eta > a/a_{\rm eq}$) are effectively massless and scale as relativistic matter $\propto a^{-4}$. The position of the peak is determined by the mass of the B-field. In fact, we have

$$k_{\rm peak} = \sqrt{H_I m}. \qquad (26)$$

Now that we know the power spectrum, we can calculate the energy density of the B-field, defined by

$$\rho_B = \int \frac{{\rm d}k}{k} P_B. \qquad (27)$$

A good dark matter candidate should have an energy density

$$\frac{\rho_B}{\rho_{\rm rad}} = 1 \qquad \text{at} \quad \eta = \eta_{\rm eq}, \qquad (28)$$

where $\rho_{\rm rad}$ is the energy density of the cosmic radiation. The calculation is done in [12] and it is found that

$$m = 2.8 \times 10^{-2}\left(\frac{10^{13}\ {\rm GeV}}{H_I}\right)^4 {\rm eV} \qquad (29)$$

gives the right energy density. Since we lack a guiding principle, there is at present no natural explanation for such a mass. Its origin might be geometrical, it might be generated by a Yukawa-type term or it could just be a 'bare' mass term.

5. Discussion and conclusion

We have shown that, while the nonsymmetric theory of gravitation is an extremely interesting extension of general relativity to study, the modes of the anti-symmetric metric field are unstable. This instability manifests itself through a wrong sign in front of spatial derivatives in the equations of motion. Such a wrong sign means that large momenta are no longer suppressed, and therefore the field grows without bounds. We showed that the troublesome terms in the Lagrangian (2) are the coupling to the Riemann tensor and the Ricci tensor. Furthermore in [5] it was shown that the first of these terms cannot be removed in NGT and that the instabilities are not a relic of the linearization. However, our linearization procedure was rather naive and it lacks a good guiding principle. Our analysis shows that *if* one could find a good principle from which to construct a nonsymmetric theory of gravitation (e.g. by considering complex manifolds as in [2, 6]), the linearized Lagrangian *must* have the form of (18). Based on this knowledge, we have studied the evolution of quantum fluctuations, generated at inflation, throughout the cosmological history. We find that the B-field has the right energy density to fully take account for the dark matter energy density if the mass of the field is given by $m = 2.8 \times 10^{-2}\left(\frac{10^{13}\,\text{GeV}}{H_I}\right)^4$ eV. Furthermore, the power spectrum develops a characteristic peak, that for this mass and $z = 10$ (start of structure formation) has a length scale coincidentally corresponding to the earth–sun distance. Although the mass of the B-field is small (equivalent to the mass of the τ-neutrino), it is still *cold* dark matter. Indeed, since the field does not couple to matter fields, it cannot thermalize and therefore the spectrum stays primordial and highly non-thermal. Because of this, it does not suffer from the problems that neutrino dark matter has. If the B-field indeed is the physical dark matter, gravity may get modified at scales $m^{-1} \propto 0.1\,\mu\text{m}\left(\frac{H_I}{10^{13}\,\text{GeV}}\right)^4$. This is still about two orders of magnitude below the current experimental bound [13].

As a final remark, we would like to stress that nonsymmetric gravity is a very interesting and natural extension of general relativity, with promising results on the issue of dark matter. However, for the theory to be viable, it is imperative to find a guiding principle that naturally leads to the stable Lagrangian (18).

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